TABLE I
Admittance of Coupled V Antennas (millimhos)

s/λ	Y_{11}	Y_{12}	Y_{22}
0.1	$17.99 - j43.91$	$-10.25 + j45.60$	$22.31 - j48.48$
0.2	$14.19 - j14.40$	$-2.26 + j15.72$	$14.65 - j14.95$
0.3	$14.34 - j6.74$	$0.58 + j9.56$	$14.50 - j6.89$
0.4	$16.06 - j3.09$	$2.83 + j7.17$	$16.13 - j3.22$
0.5	$18.98 - j1.80$	$5.47 + j4.90$	$18.95 - j1.92$
0.6	$21.39 - j3.62$	$7.13 + j0.61$	$21.28 - j3.64$
0.7	$20.39 - j5.97$	$5.01 - j3.62$	$20.35 - j5.90$
0.8	$18.59 - j5.72$	$1.58 - j4.79$	$18.63 - j5.67$
0.9	$18.26 - j4.48$	$-1.04 - j4.23$	$18.31 - j4.49$
1.0	$19.03 - j3.80$	$-2.93 - j2.79$	$19.04 - j3.84$

$a/\lambda = 0.007022$, $h/\lambda = 0.25$, $\psi = 90°$.

TABLE II
Admittance of Coupled V Antennas (millimhos)

s/λ	Y_{11}	Y_{12}	Y_{22}
0.1	$6.77 - j17.91$	$-0.23 + j17.94$	$7.22 - j18.93$
0.2	$7.53 - j7.54$	$0.95 + j8.08$	$7.64 - j7.72$
0.3	$8.77 + j4.10$	$1.95 + j5.37$	$8.83 - j4.19$
0.4	$10.57 - j2.54$	$3.25 + j3.86$	$10.59 - j2.63$
0.5	$12.71 - j2.77$	$4.71 + j1.71$	$12.65 - j2.83$
0.6	$13.30 - j4.73$	$4.47 - j1.59$	$13.23 - j4.71$
0.7	$11.88 - j5.53$	$2.03 - j3.37$	$11.88 - j5.48$
0.8	$11.06 - j4.79$	$0.15 - j3.23$	$11.09 - j4.77$
0.9	$11.28 - j4.00$	$-1.60 - j2.38$	$11.31 - j4.02$
1.0	$11.95 - j3.88$	$-2.49 - j1.06$	$11.95 - j3.90$

$a/\lambda = 0.007022$, $h/\lambda = 0.25$, $\psi = 120°$.

ten segments, and a piecewise-sinusoidal expansion is employed for the current function. As described in [3], the integral equation is thus reduced to a system of simultaneous linear equations. The general theory of these moment methods is presented by Harrington [4].

Numerical Results

Figs. 1 and 2 show the mutual admittance $Y_{12} = G_{12} + jB_{12}$ versus the spacing s for an array of half-wave V antennas. When the angle ψ is 180°, it may be noted that the results agree with those of Chang and King [2] for parallel linear antennas.

Tables I and II show the self- and mutual admittances versus spacing for $\psi = 90°$ and 120°.

Using an IBM 7094 computer, 5 seconds are required to solve a two-element array problem with off-center terminals. Each wire antenna may have arbitrary shape, position, and orientation. Optional output data includes the radiation efficiency and the gain $G(\theta,\phi)$. For example, Yagi and log-periodic V arrays are readily analyzed.

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Ray-Diffraction Method for Handling Complex Incident Fields

***Abstract*—A ray-diffraction method is described which is applicable to diffraction problems where the incident field on a diffracting edge cannot be represented as a single incident ray but exhibits a more complex form. The complex incident field may be represented by a combination of simple rays; the diffraction of each ray may then be treated individually. The total diffracted field is the sum of these diffraction components. The method is illustrated by applying it to the double knife-edge problem.**

I. Proposed Method

Ray-diffraction techniques [1]–[8] provide solutions to problems which are difficult to solve by other approaches. This communication proposes a method for applying ray optics to diffraction problems involving complex incident fields on an edge which cannot be described simply as emanating from a point or line source.

The proposed method consists of determining a set of simple sources which produce the same amplitude and phase characteristics as those of the actual incident field in the near neighborhood of a diffracting edge. The diffraction coefficients can then be calculated individually for each source, and the resultant diffracted field from the edge may be determined by superposition. To describe the details of the method, it is applied to the double knife-edge problem, where the complex field on the shadow boundary of the edge closest to the source is incident on the far edge.

II. Double Knife-Edge Example

Fig. 1 shows the geometry for the double knife-edge problem. The line source and the two knife edges are parallel and normal to the paper. Three sets of polar coordinates are defined with the origins at the line source and the two edges.

The incident field on the near edge is a cylindrical wave and can be represented by a single ray emanating from the line source. The incident field (any polarization) on the far edge, near the shadow boundary, can be shown to be given approximately by [9], [10]

$$G(u) = [\tfrac{1}{2} + (2/\pi)^{1/2}u \exp(j\pi/4)] \exp(j2u^2) \tag{1}$$

where $G(u)$ is the total field relative to field on the shadow boundary ($\theta = 0$) at r, when the near edge is removed, $u = [kbr/(b + r)]^{1/2} \times \sin(\theta/2)$, $k = 2\pi/\lambda$, and λ is the free-space wavelength.

Equation (1) describes the incident field near the far edge when $|u| < 0.5$. It corresponds to a single diffraction from the near edge and does not include multiple diffractions between the two edges. The multiple diffractions are usually negligible in such problems as long as $b < c$. It is also assumed that the time dependence is $\exp(j\omega t)$, and that kb and $kr \gg 1$.

The objective now is to find a combination of simple line sources whose incident field on the far edge approximates, in both amplitude and phase, the incident field as represented by (1). Of primary importance is the location and the excitation of the sources. The location of the sources can be determined by a consideration of the phase curvature of the incident field near the far edge. This is given by the factor $\exp(j2u^2)$ in (1) where

$$2u^2 = 2kbc/(b + c) \sin^2(\theta/2). \tag{2}$$

It can be shown, by simple geometry, that this phase characteristic corresponds to any source having a phase center located at the original source point.

Fig. 2 shows a line-source array located with its phase center at the original source point, satisfying the preceding phase requirement, and also providing the means for obtaining the proper amplitude

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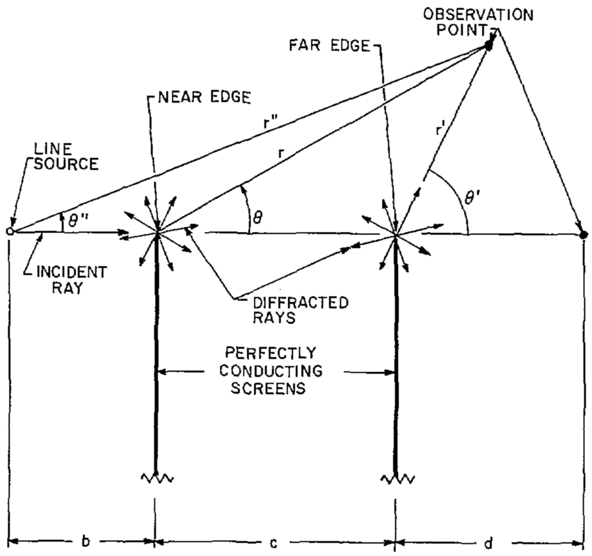

Fig. 1. Double knife-edge geometry.

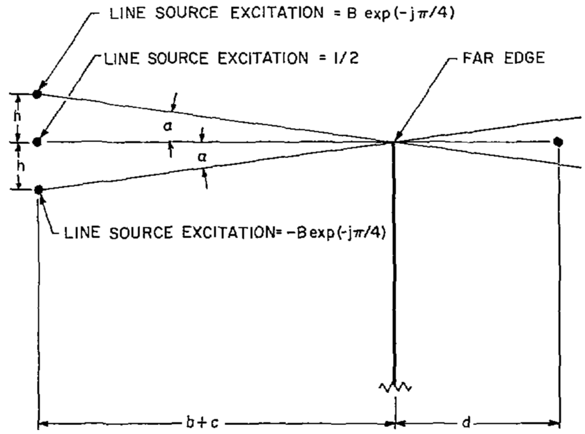

Fig. 2. Virtual sources providing proper incident field on far edge

in the near neighborhood of the far edge. The far-field pattern of the line-source array is given by

$$A(\theta'') = \tfrac{1}{2} + 2B \sin\left[(2\pi h/\lambda) \sin \theta''\right] \exp(j\pi/4). \tag{3}$$

The 1/2 factor on the right corresponds to the center line source. The other factor corresponds to the remaining doublet. In (3), B is the excitation amplitude of each line source of the doublet. For small angles it can be shown that the amplitude and phase pattern for the line-source array, as given by (2) and (3), can be made identical to the amplitude and phase of the actual incident field as given by (1).

In order to evaluate B, the right side of (3) is equated to the amplitude term in (1). This results in

$$2B \sin\left(\frac{2\pi h}{\lambda} \sin \theta''\right) = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{kbc}{b+c}\right)^{1/2} \sin\frac{\theta}{2}. \tag{4}$$

Since the value of h has not been specified, it is now selected so that the sine functions can be replaced by their arguments. Thus h is restricted by

$$\frac{2\pi h}{\lambda} \sin \theta'' < \frac{\pi}{16}.$$

This is satisfied if $h = \lambda/32$; moreover, since θ and θ'' are small, B is now determined as

$$B = \frac{4}{\pi}\left[\frac{2}{\pi}\frac{kb}{c}(b+c)\right]^{1/2}. \tag{5}$$

Thus the virtual source arrangement shown in Fig. 2, with the amplitude and phase as determined previously, provides a field that closely approximates the actual amplitude and phase of the incident field in the neighborhood of the far edge. Each line source can now be considered as exciting diffracted rays off the far edge; the diffraction coefficients or patterns for each can be determined from the basic solution of diffraction by an edge excited by a single line source [9], [10]. As an example of this procedure, the case of the field at the shadow boundary of the two edges is calculated.

The field is to be determined at the point $r' = d$ and $\theta' = 0$. In Fig. 2, α is shown to be the angle separating the shadow boundaries associated with the virtual line sources. Since α is very small, it is possible to use (1) for evaluating the field on the shadow boundary attributed to each line source. For the center line source, $u = 0$; for the doublet line sources, u is given by

$$u = \left[\frac{k(b+c)d}{b+c+d}\right]^{1/2} \sin \pm\frac{\alpha}{2} \tag{6}$$

$$|u| = \frac{\lambda}{64}\left[\frac{kd}{(b+c+d)(b+c)}\right]^{1/2}. \tag{7}$$

The magnitude of u can be shown to be less than 0.5. The total field G is given by the superposition of three components:

$$G \approx \tfrac{1}{4} + \left[\tfrac{1}{2} + (2/\pi)^{1/2} |u| \exp(j\pi/4) - \tfrac{1}{2} + (2/\pi)^{1/2} |u| \exp(j\pi/4)\right] B \exp(-j\pi/4)$$

$$G = \tfrac{1}{4} + 2B(2/\pi)^{1/2} |u|. \tag{8}$$

The quadratic phase term in (1) is assumed to be unity (i.e., only the leading term is considered). Substituting (5) and (7) into (8) results in

$$G = \frac{1}{4} + \frac{1}{2\pi}\left[\frac{(b/c)(d/c)}{1 + b/c + d/c}\right]^{1/2}. \tag{9}$$

This result closely approximates the field at the shadow boundary under the restrictions that kb, kc, and $kd \gg 1$ and $b/c < 1$. As in previous cases, where ray-optical results were compared with results obtained by other techniques, this result is also in agreement with one obtained by means of a technique which utilized a double surface integration to determine the field [11].

It is interesting to note that, as in the case of the 6-dB decrease in field strength along the shadow boundary for a single knife edge, the field on the shadow boundary in this case is also independent of frequency and is simply a function of geometry. It is also noted that (9), as required, satisfies reciprocity with respect to the source and observation points. Fig. 3 shows G plotted as a function of d/c with b/c as a parameter. It is observed that the minimum relative field is down 12 dB and that the relative field strength increases as the observation point is further removed from the far edge.

This example illustrates how a complex incident field on a diffracting obstacle can be decomposed so that ray-optical techniques

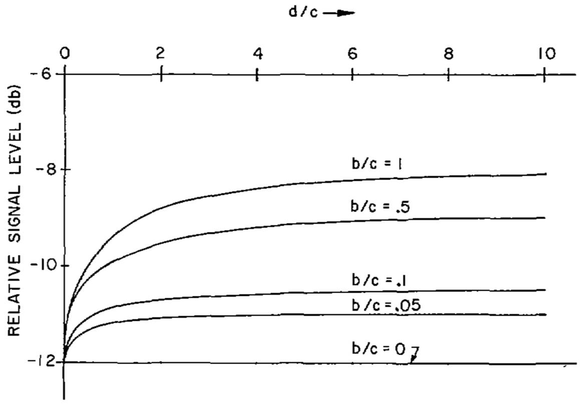

Fig. 3. Total signal along shadow boundary relative to signal at same observation point when both edges are removed.

can be applied to the simpler constituent parts. This procedure can provide solutions to problems which would be difficult by other techniques.

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Scattering Cross Section for Two Cylinders

Abstract—**Analytical expressions are derived for the total scattering cross section of two identical parallel nonoverlapping circular cylinders in the immediate neighborhood of each other. Results obtained for the limiting case of Rayleigh particles are compared with the corresponding results obtained with the Rayleigh–Gans approximation.**

I. Introduction

Most of the authors [1]–[5] who had worked on solving problems of multiple scattering from a collection (random or regular) of parallel cylinders have used some form of perturbation or approximation method or have considered or imposed special conditions on the parameters of the particles. (Other less pertinent work can be traced from the reference list.)

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In examining the case of an arbitrary configuration of parallel cylinders, Twersky [1] employed an iterative procedure as follows. The independent (single cylinder, or first order) scattering waves of the nth cylinder are taken as secondary incident waves on the νth cylinder ($n \neq \nu$) thereby obtaining a second-order approximation for the scattered waves of the system. The process is then repeated using the second-order scattered waves as the secondary incident waves to obtain a third-order approximation to the true scattered waves, and so on. For convergence of the iteration to be readily ensured, one requires that successive contributions should decrease with the increase in the number of steps of the iteration; if not errors may accumulate. His results appear therefore to be most useful for special cases of the problem, e.g., Rayleigh particles, widely separated or soft particles.

Row [2] has considered the scattering from an arbitrary array of parallel cylinders in general and two identical conducting cylinders in particular. It is evident that his method can be readily generalized to cover the case of dielectric particles. Using a special diagonal approximation method, he obtained a reasonable estimate of the vector potential of the ensemble which he used to present results for cases where the separation is generally not less than six times the particle radius. Other investigations [6]–[8] have shown that for small particles of the type considered by Row, the multiple scattering effects are usually small whenever the separation is more than four times the particle radius. Consequently, doubts exist as to the suitability of the approximation at contact positions or about this region of separation.

The scattering of a plane wave by a row of small cylinders was treated by Millar [3]. This was another case of solution by approximation methods. Zitron and Karp [4] and Karp [5] also assumed large separation between particle pairs in arriving at their solutions. Recently, however, an exact solution was given [9] for the problem of scattering by two neighboring circular cylinders. This solution can be readily extended to any collection of cylinders of other cross sections provided that the wave equation is separable for any member of the ensemble. No restrictions on the geometry or composition have been imposed except that the cylinders are placed normally to the direction of propagation of a plane plane-polarized electromagnetic wave.

In this paper we would like to give the derivation of the general formula for the total scattering cross section from two identical parallel circular cylinders with complex index of refraction and use the formula to estimate the validity of the independent scattering approximation for Rayleigh particles. Comparison is made with the results of Lillesaeter [7] for the case of two Rayleigh spheres, which he computed using Trinks' solution [6].

II. Scattering Cross-Section Formula

Consider two identical parallel circular cylinders, of radius a and refractive index m, placed in the field of a plane plane-polarized electromagnetic wave such that the plane containing the axes of the cylinders makes an angle β with the direction of propagation of the incident wave. Denote by r,θ the polar coordinates of an observation point P with respect to the center of one cylinder (Fig. 1). Then the total scattering cross section $C_{sca}^{(2)}$ may be taken as

$$Q_{sca}^{(2)} = C_{sca}^{(2)}/2\pi a = \frac{1}{\pi x}\int_0^{\pi} |T_2(\theta)|^2 \, d\theta \quad (1)$$

where if $\delta = kd$, d being the separation distance between the cylinders, then

$$T_2(\theta) = \sum_{n=-\infty}^{\infty} \{{}_1b_n + {}_2b_n \exp(-i\delta\cos\theta)\} \exp(-in(\theta+\beta)) \quad (2)$$

is the scattering amplitude function for two cylinders [10], when the normally incident wave is polarized parallel to the axes