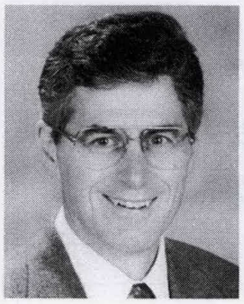

Tom Milligan
Milligan & Associates
8204 West Polk Place
Littleton, CO 80123
(303) 977-7268
(303) 977-8853 (Fax)
TMilligan@ieee.org (e-mail)

Wheeler and Fano Impedance Matching

Alfred R. Lopez

ARL Associates
4 Sarina Drive, Commack, NY 11725 USA
Tel: +1 (631) 499-2987; Fax: +1 (631) 462-0320; E-mail: al.lopez@ieee.org

Abstract

A recent series of articles in this *Magazine*'s Antenna Designer's Notebook column highlighted Fano's fundamental work on the limitations of the impedance matching of arbitrary impedances and its application to antenna design. Lopez, using Harold Wheeler's methodology, constructed a closed-form solution to the nonlinear simultaneous equations of Fano. This article presents the Wheeler methodology that was the basis for this work, and is still invaluable in the understanding of optimum impedance matching.

Keywords: Impedance matching; narrow-bandwidth antennas; Q factor

1. Introduction

This article presents some of Harold A. Wheeler's early work (from the 1940s) on impedance matching [1]. It presents the development of his formulas for optimum impedance matching using single- and double-tuned matching circuits. His equations relate the antenna Q-bandwidth (QB) product to the maximum reflection magnitude (Γ) over the frequency bandwidth $B = (f_H - f_L)/\sqrt{f_H f_L}$, where f_H and f_L are the high- and low-edge-band frequencies. During this same period of time, Robert M. Fano [2] developed his famous impedance-matching equations, which relate the Q-bandwidth product of a load to the maximum reflection magnitude for all levels of multiple tuning circuits. Lopez [3] later showed that the Wheeler and Fano equations were in exact agreement for the Fano $n = 1$ and $n = 2$ cases. This article was motivated by the series of articles on "Fano matching" that recently appeared in this *Magazine*'s Antenna Designer's Notebook column [4-8].

2. Wheeler Single-Tuned Edge-Band Matching

Figure 1 shows the circuit diagram for series single tuning of an antenna. Wheeler started with a reference case, which he referred to as the single-tuned mid-band match case. The Smith-chart impedance locus for this case is shown in Figure 2. The ideal transformer shown in Figure 1 is set so that a match (zero reflection) is achieved at the resonant frequency ($f_0 = \sqrt{f_H f_L}$). This case does not yield the minimum maximum-reflection magnitude (Γ) at the band edges.

Wheeler next demonstrated that the minimum Γ is achieved when the edge-band frequencies lie on the vertical axis of the Smith chart. Impedance transformation is used to achieve this condition, as shown in Figure 2. The ratio of the generator resistance, R_G, to the resonant resistance, R_{EB}, for the edge-band matching case is given by

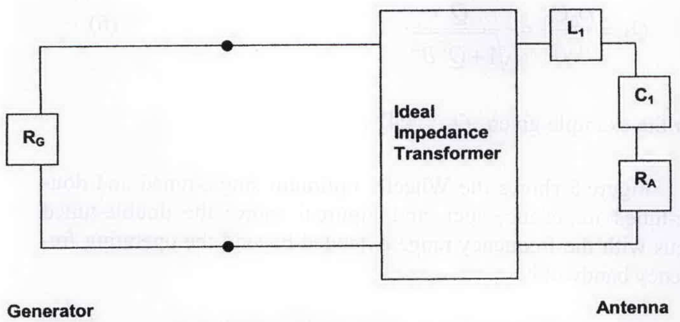

Figure 1. A single-tuned antenna impedance-matching circuit.

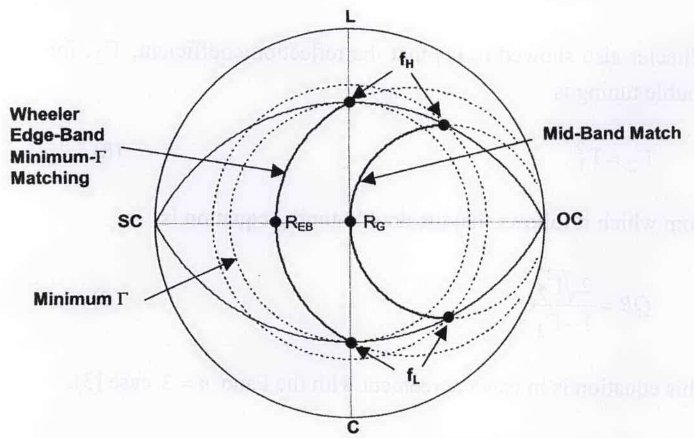

Figure 2. Wheeler optimum single-tuned impedance matching.

$$\frac{R_G}{R_{EB}} = \sqrt{1+Q^2B^2}, \tag{1a}$$

where

$$\mathrm{Q} = \frac{\omega_0 L_1}{R_A}. \tag{1b}$$

For the loci shown in Figure 2, $Q=10$, $B=0.2$, and $R_G/R_{EB}=2.24$.

Wheeler [1] showed that at the edge-band frequencies for the edge-band matching case, the impedance is given by

$$Z_{EB} = \exp\left(\pm j\phi_{EB}\right) \tag{2a}$$

and

$$\tan\left(\phi_{EB}\right) = QB. \tag{2b}$$

This equation assumes that the Smith chart is normalized to unit resistance.

Wheeler also showed that

$$\Gamma = \tan\left(\phi_{EB}/2\right). \tag{3}$$

The tangent half-angle formula [3] provides an explicit relationship between the Q-bandwidth product and the maximum reflection magnitude:

$$\tan\left(\phi_{EB}\right) = \frac{2\tan\left(\phi_{EB}/2\right)}{1-\tan^2\left(\phi_{EB}/2\right)}, \tag{4}$$

from which followed

$$QB = \frac{2\Gamma}{1-\Gamma^2}. \tag{5}$$

This equation is in exact agreement with the Fano $n=1$ impedance-matching case [3].

3. Wheeler Double-Tuned Matching

Wheeler went on to show that the optimum double-tuned impedance matching was derived directly from the optimum single-tuned case. Figure 3 shows the circuit diagram for the double-tuned case. The parallel resonant circuit has the same resonant frequency (f_0).

Figure 4 is a Smith chart showing the required susceptance for the parallel resonant circuit to achieve optimum double-tuned impedance matching. The Q for the parallel resonant circuit is

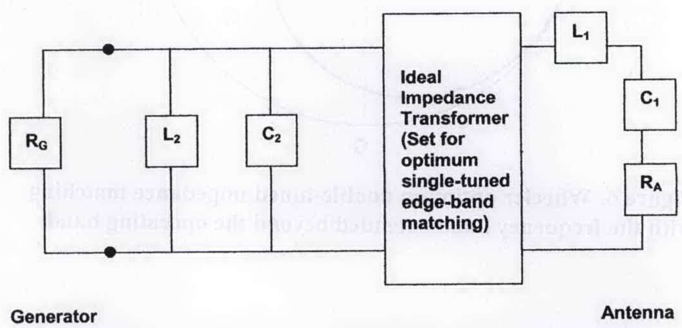

Figure 3. A double-tuned antenna impedance-matching circuit.

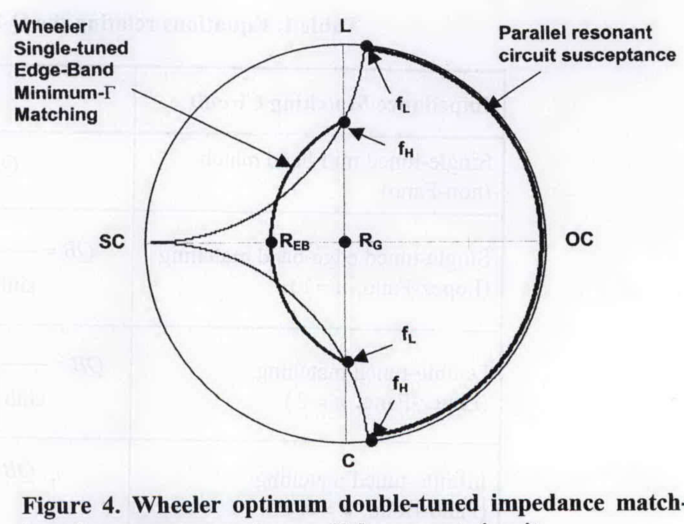

Figure 4. Wheeler optimum double-tuned impedance matching: the susceptance of a parallel resonant circuit.

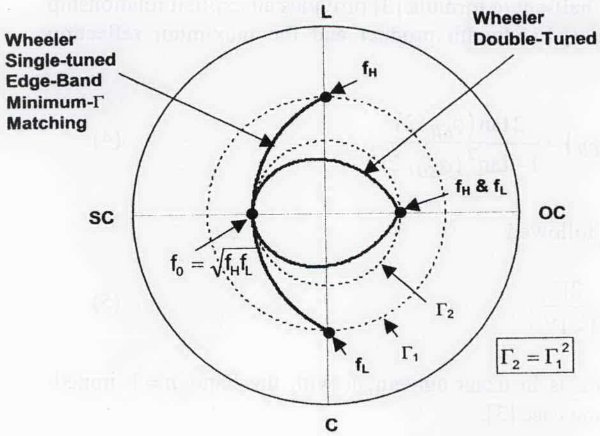

Figure 5. Wheeler optimum double-tuned impedance matching.

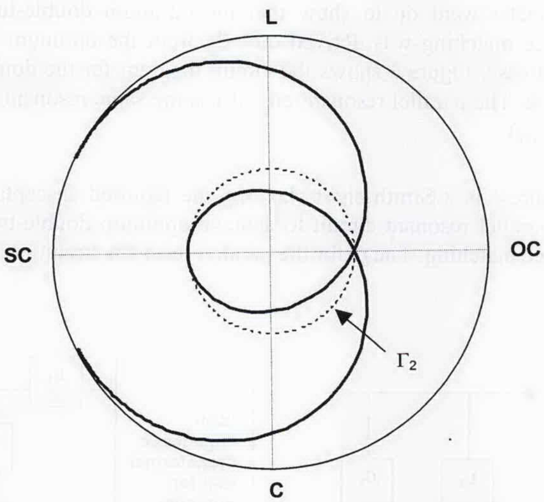

Figure 6. Wheeler optimum double-tuned impedance matching with the frequency band extended beyond the operating band.

$$Q_2 = \frac{\omega_0 C_2}{G_G} = \frac{Q}{\sqrt{1+Q^2B^2}}. \tag{6}$$

For the example given, $Q_2 = 4.47$.

Figure 5 shows the Wheeler optimum single-tuned and double-tuned impedance loci, and Figure 6 shows the double-tuned locus with the frequency range extended beyond the operating frequency bandwidth.

Equation (5) for single tuning can be rewritten as

$$QB = \frac{2\Gamma_1}{1-\Gamma_1^2}. \tag{7}$$

Wheeler also showed in [1] that the reflection coefficient, Γ_2, for double tuning is

$$\Gamma_2 = \Gamma_1^2, \tag{8}$$

from which it follows that the double-tuning equation is

$$QB = \frac{2\sqrt{\Gamma_2}}{1-\Gamma_2}. \tag{9}$$

This equation is in exact agreement with the Fano $n = 2$ case [3].

4. Summary

Table 1 presents Wheeler's equations and the equation for the single-tuned mid-band match case. The equations include a conversion from the maximum reflection magnitude (Γ) to the maximum VSWR (V). Also included in the table is the Fano-Bode case for an infinite number of tuned circuits. From Table 1 and for the given VSWR, it is noted that the single-tuned edge-band matching

Table 1. Equations relating the Q-bandwidth product to the VSWR.

Impedance Matching Circuit	Equation	QB for V = VSWR = 2
Single-tuned mid-band match (non-Fano)	$QB = \frac{2\Gamma}{\sqrt{1-\Gamma^2}} = \frac{V-1}{\sqrt{V}}$	$QB = 0.707$
Single-tuned edge-band matching (Lopez-Fano, $n = 1$)	$QB = \frac{1}{\sinh\left[\ln\left(\frac{1}{\Gamma}\right)\right]} = \frac{2\Gamma}{1-\Gamma^2} = \frac{V^2-1}{2V}$	$QB = 0.750$
Double-tuned matching (Lopez-Fano, $n = 2$)	$QB = \frac{1}{\sinh\left[\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)\right]} = \frac{2\sqrt{\Gamma}}{1-\Gamma} = \sqrt{V^2-1}$	$QB = 1.732$
Infinite-tuned matching (Fano-Bode, $n = \infty$)	$QB = \frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)} = \frac{\pi}{\ln\left(\frac{V+1}{V-1}\right)}$	$QB = 2.860$

case provides a small benefit over the single-tuned mid-band matching case. It is also seen that for a given Q, the double-tuned matching case provides an increase of more than double the bandwidth of the single-tuned mid-band matching case for a $\mathrm{VSWR}=2$. In theory, an infinite number of tuning circuits will only provide a two-thirds increase over the double-tuned matching case.

The double-tuned matching case is commonly used for impedance matching. This case provides a good measure of the bandwidth that can be achieved for the impedance matching of a narrow-bandwidth antenna. As has been pointed out [1-7], there is a law of diminishing returns for Fano matching beyond the $n=2$ case. In practice, one should not expect to achieve a bandwidth that is much larger than that achieved with the double-tuned matching case. A simple rule of thumb for the achievable bandwidth for a $\mathrm{VSWR}>2$ is that it is approximately equal to the VSWR divided by the Q.

5. Comment on the Wheeler and Fano Approaches

Wheeler had an ability for reducing complex scientific principles to simple forms that were universally helpful to theoreticians and practitioners. His work on impedance matching is another good example of this ability. He developed the art of impedance matching using the reflection chart as the primary tool.

The contrast between the Wheeler approach and the Fano approach is interesting. Fano, for the most part, relied heavily on mathematical rigor; Wheeler, on the other hand, reduced the problem to a form where the solution was apparent by simple geometrical considerations.

Fano's approach was comprehensive. It provided a complete picture of the basic limitations for the impedance matching of arbitrary impedances. In retrospect, the community benefits from both the Wheeler and Fano contributions.

6. Acknowledgement

Many thanks are given to Peter W. Hannan for his critical review of this article and for his many helpful comments and suggestions. Pete is credited with the simple rule of thumb for the achievable bandwidth.

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